Lesson 34

Numbers and Indices

Aims

The aim of this lesson is to enable you to:

- work with rational and irrational numbers
- work with surds to rationalise the denominator
- when calculating interest, tackle reverse percentage problems and negotiate combinations of increases and decreases
- consider negative and fractional powers and indices
- calculate direct and inverse variation
- identify sequences where part of the formula is n^2

Context

In this section of the course, we tackle some more advanced topics. Often they are an extension of material you have already covered earlier in the course. You may need to remind yourself of the skills learnt in Lessons One, Four, Ten and Thirteen.

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Working with Irrational Numbers

As you know, an **irrational number** is one which cannot be expressed as a fraction, e.g. the square root of 3, written as $\sqrt{3}$. A calculator will show it in decimal form as 1.7320508 …… i.e. it never terminates or repeats itself. π (pi) is another example of an irrational number. π = 3.141592 …..

A **surd** is an irrational number that *contains* a square root, e.g.

$2\sqrt{3}$ $\quad \frac{2}{\sqrt{3}}$ $\quad \sqrt{3}\times\sqrt{7}$ $\quad 10 - \sqrt{3}$ $\quad 4 + 3\sqrt{3}$ etc

Sometimes surds can be simplified. There are two rules which enable simplification to take place.

Consider, for example, $\sqrt{3}\times\sqrt{7}$. There are two operations involved here: firstly, square rooting each number, and secondly, multiplying. Now it turns out that these two operations can be performed in the reverse order, so that the two numbers are first multiplied and then the square of the result is found:

$$\sqrt{3}\times\sqrt{7} = \sqrt{3\times 7} = \sqrt{21}$$

If you try both versions on a calculator you should get the same result: 4.582575695…

The strategy for simplifying a surd can be summarised as: Express the number under the root sign as a product of two numbers. This is called a factor pair, one of which is a **square** number. You are looking for the *largest* square number factor.

Use the above rule to reverse the order of the two operations.

Log on to Twig and look at the film titled: **Irrational Numbers**

www.ool.co.uk/1746gc

Beside the now-famous theorem which bears his name, discover the bizarre life of Pythagoras and his Brotherhood, in which maths was the language of the Gods and irrational numbers were heretical.

Example

Express each of the following as the 'simplest possible surd':

(a) $\sqrt{8}$ **(b)** $\sqrt{75}$ **(c)** $\sqrt{48}$

The aim is to express 8 as the product of two numbers, one of which is a square number. In fact, the only way to split 8 into a product is as 4×2, and we see that one of the numbers (4) is a square number. So $\sqrt{8} = \sqrt{4 \times 2}$. Reversing the order of operations, we get $\sqrt{4} \times \sqrt{2}$. But we know that $\sqrt{4}$ is simply 2. We can therefore conclude that $\sqrt{8}$ can be written as $2 \times \sqrt{2}$, although it is usually written as $2\sqrt{2}$. (When no operation is indicated, it must be multiplication.) If you check on a calculator, you should find that $2 \times \sqrt{2}$ and $\sqrt{8}$ are both 2.828427125…

Try dividing 75 by each of the square numbers in turn, until one goes exactly without remainder. We do not bother with the first square number, 1. There are remainders when 75 is divided by 4, 9 and 16. However, there is no remainder when dividing 75 by 25: $75 \div 25 = 3$.

So $\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5 \times \sqrt{3}$ or even more simply as $5\sqrt{3}$.

48 is divisible by the second square number, 4. So we could proceed to simplify $\sqrt{48}$ as $\sqrt{4 \times 12}$. However, it is much more efficient to find the *largest* square number which divides into 48. In fact, the fourth square number (16) divides into 48.

So $\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = 4 \times \sqrt{3}$ or simply $4\sqrt{3}$.

Activity 1

Express each of the following as the 'simplest possible surd':

(a) $\sqrt{18}$ (b) $\sqrt{20}$ (c) $\sqrt{27}$ (d) $\sqrt{32}$

(e) $\sqrt{72}$ (f) $\sqrt{80}$ (g) $\sqrt{98}$ (h) $\sqrt{216}$

Surds: Rationalising the Denominator

We have already looked at ways of rationalizing (or simplifying) the denominator of a fraction.

A surd is an irrational root or an expression containing irrational numbers or complex numbers, e.g. $\sqrt{2}$ + 5k is a surd.

Consider the fraction $\frac{4}{\sqrt{5}}$. Is there any way of simplifying it?

Often it is helpful to "rationalize the denominator", i.e. turn the denominator into a rational number. How do we do that? By multiplying top and bottom by a quantity which would enable us to eliminate the square root sign. In this example, we can multiply top and bottom by $\sqrt{5}$:

$$\frac{4}{\sqrt{5}} = \frac{4}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{4\sqrt{5}}{5}$$

Activity 2	Rationalize the denominator in the following:
	(a) $\frac{1}{\sqrt{10}}$ (b) $\frac{3}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2\sqrt{5}}$

Reverse Percentage Problems

We now extend the idea of 'growth rate', although it would be better to use the more general term 'multiplier' since we are about to consider things that reduce. Recall the rule for finding the 'multiplier' when something increases by 5%:

- Convert 5% into a decimal: 5% = $\frac{5}{100}$ = 0.05

- Add the decimal to 1: 1 + 0.05 = 1.05

Now, if something were to reduce by 5%, then the only difference is in the second line: instead of adding the decimal, we subtract the decimal. So to reduce something by 5%, the quick way is to multiply by 0.95 (since 1 – 0.05 = 0.95).

Make sure you understand the following table.

%	**Multiplier for:** an increase	**Multiplier for:** A decrease
2	1.02	0.98
4.1	1.041	0.959
15	1.15	0.85
36.9	1.369	0.631

The 'multiplier' for repeated reductions can be used in the same way as for an increase. 'Depreciation' is one such context. If something depreciates at 10%, then it loses 10% of its value each year. This is like compound interest in reverse, so the same technique applies:

Final value = original value × multiplier $^{\text{years}}$

This powerful formula works for both increases and decreases.

Example

A car depreciates by 15% each year. If it was worth £12000 when new, what will it be worth after five years?

Convert 15% to a fraction: 15% = $\frac{15}{100} = 0.15$. Now subtract from 1 for a reduction: 1 – 0.15 = 0.85 .

The value after five years will be $12000 \times 0.85^5 = 5324.46$.

Reverse Calculations

We are sometimes required to work 'backwards': given the value **after** an increase/decrease find the **original** value. Look again at the formula for obtaining the final value after an increase/decrease:

Final value = original value × multiplier $^{\text{years}}$

The key detail is the multiplication sign. To reverse the calculation we perform the opposite of multiplication: division.

Table 1 concerns four situations which are all 'forward', we are given the original amounts, and need to find the final amount after a percentage increase or decrease. Notice that the multipliers are greater than one for the increases, and less than one for the decreases.

Table 1

Original amount	%	Increase/ decrease?	Multiplier	Final amount
200	5	Increase	1.05	200×1.05 = 210
5300	7.5	Decrease	0.925	5300 × 0.925 = 4902.50
480	2.3	Increase	1.023	480 × 1.023 = 491.04
9654	3	Decrease	0.97	9654 × 0.97 = 9364.38

In contrast, Table 2 concerns situations which are all 'backward'. We are given the **final** amount after a certain percentage increase or decrease, and we need to work 'backward' to find the **origina**l amount. Again, notice that the multipliers are greater than one for the increases, and less than one for the decreases.

Table 2

Final amount	%	Increase or decrease?	Multiplier	Original amount
731	4.1	Increase	1.041	731 ÷1.041 = 702.21
9601	6.8	Decrease	0.932	9601 ÷ 0.932 = 10301.50
554.78	9.5	Increase	1.095	554.78 ÷1.095 = 506.65
17439.07	13.2	Decrease	0.868	17439.07 ÷ 0.868 = 20091.09

Concentrate on the one important difference between Tables 1 and 2. In Table 1 the calculations in the right hand columns are all multiplications (because we are working 'forwards'); in Table 2 they are all divisions (we are working 'backwards').

Combinations of Increases and Decreases

Example

An investment increases in value by 3% in the first year, 0.5% in the second year, and decreases by 1.5% in the third year. What is the value of the investment at the end of the three years if the initial amount was £1000?

The multipliers for the three years are 1.03, 1.005 and 0.985 respectively. We simply multiply the original amount by each of the multipliers: 1000 × 1.03 × 1.005 × 0.985 = 1019.62

Activity 3

Work out the following. You may use a calculator:

1. A sum is invested in a bank at a fixed rate of 10% annual compound interest. After 4 years, the sum has grown to £1,317.69. What was the original sum?

2. A sum is invested in a bank at a fixed rate of 6% annual compound interest. After 5 years, the sum has grown to £3345.56. What was the original sum?

Activity 4

1. Copy and complete the following table:

	Original amount	%	Increase or decrease?	Multiplier	Final amount
a	£100	2.7	Increase		
b	£6480	3.82	Decrease		
c	£792	12.6	Increase		
d	£5000	8.47	Decrease		
e		1.9	Increase		£635.23
f		4.25	Decrease		£7019.44
g		5.83	Increase		£74.80
h		14.6	Decrease		£3292.73

2. A special investment account pays a fixed compound interest of 7% per annum. Money can neither be added nor withdrawn during the first three years of the account.

   a) If £2500 is paid in to the account at the beginning, how much will it be worth at the end of the first three years?
   b) Suppose you need the account to be worth £5000 at the **end** of the first three years. What must be the value of your **initial** deposit?

3. A new car costs £19950. It depreciates by 25% in the first year, 20% in the second year and 15% in the third year. How much will the car be worth after the three years?

4. An investment increases in value by 8% in its first year. It decreases in value by 8% in the second year. If the initial value of the investment is £1000, what is the value of the investment after the two years? Are you surprised?

5. An investment increases in value by 11% in its first year. It decreases in value by 10% in the second year. If the initial value of the investment is £5000, what is the value of the investment after the two years? Are you still surprised?

Negative and Fractional Powers

You should recall how to find square roots and cube roots. We now introduce some new notation. In general, the new notation is summed up as follows:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

This simply means that if a power is a fraction with numerator one, then this means the *nth root* of the number. Just get used to the new notation by looking carefully at the following examples. The calculations are revision (i.e. changing the 'root form' into the 'simplest possible form'). Concentrate particularly on the equivalence of the first two columns which say the same thing in two different ways.

Fractional Index form	**Root form**	**Simplest possible form**
$9^{\frac{1}{2}}$	$\sqrt{9}$	3
$8^{\frac{1}{3}}$	$\sqrt[3]{8}$	2
$16^{\frac{1}{4}}$	$\sqrt[4]{16}$	2
$100000^{\frac{1}{5}}$	$\sqrt[5]{100000}$	10

Note that if there is no little number to the left of the root sign, then this means a square root: in other words, a two is implied. So, for example, $\sqrt{9}$ means $\sqrt[2]{9}$.

We now need to consider the case when the numerator of a fractional index is not one. We will need a technique from Lesson One. One of the Index Laws said that if there are powers inside and outside a bracket, then these powers can be multiplied. For instance:

$$\left(2^3\right)^4 = 2^{3\times 4} = 2^{12}$$

Consider, for example, $8^{\frac{2}{3}}$. We can think of the fraction $\frac{2}{3}$ as $\frac{1}{3}\times 2$. This may seem a strange thing to do, but it enables us to use the above Index Law:

$$8^{\frac{2}{3}} = 8^{\frac{1}{3}\times 2} = \left(8^{\frac{1}{3}}\right)^2$$

But we already know that $8^{\frac{1}{3}}$ means the cube root of 8, and has the value 2. We can therefore do $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$.

Let us reflect on the parts that the numerator and denominator of the fraction have played. The denominator is three, and we found a cube root. The numerator is two and we found a square (i.e. the power two). This is always the case.

It turns out that we could even reverse the order of the two operations. If we returned, for example, to the case of $8^{\frac{2}{3}}$, but squared first, we would obtain 64. If we calculated the cube root of 64 we would obtain 4, the same as before.

However, it is usually much easier to perform the root first. Consider the following Example.

Example 1

Simplify $16^{\frac{3}{4}}$

The numerator of the fraction is three: this means that we need the cube (i.e. a power three). The denominator of the fraction is four: this means that we need a fourth root. If we do the fourth root first, we obtain 2, (since $2\times 2\times 2\times 2 = 16$). Now find the cube of 2, which is 8.

However, look what happens if we reverse the order of the two operations. Do the cube first: $16^3 = 4096$. We now need to find the fourth root of 4096. You need to know simple roots of small numbers. You should be reassured that you are **not** expected to carry with you at all times the information that the fourth root of 4096 is 8.

Of course, a calculator would help. However, you need to be able to simplify expressions like that in Example 1 **without a calculator**. So, to sum up, it is wiser to perform the root first and the power afterwards: this usually makes the mental arithmetic **much** easier.

Negative Fractional Indices

Remember that a negative power means that we mentally remove the minus sign and then do the **reciprocal** of the result. This also works for negative fractional indices, if we keep calm and proceed one step at a time.

Example 2

Simplify $32^{-\frac{3}{5}}$

Mentally remove the minus sign: $32^{\frac{3}{5}}$. The five means that we need the fifth root of 32, which is 2 (since $2 \times 2 \times 2 \times 2 \times 2 = 32$). The three means that we need the cube of 2, which is 8. Now we need to do the reciprocal of $32^{\frac{3}{5}}$, in other words $\frac{1}{32^{\frac{3}{5}}}$. The bottom of the fraction is 8, so our final answer is $\frac{1}{8}$.

Fractional Indices on a Calculator

This deserves a separate section. There are various ways of finding a fractional power of a number on a modern scientific calculator. It is suggested that you try all the following techniques. However, then decide on a method that you like, that you will remember and that works reliably for you.

Example 3

Use a calculator to evaluate $4913^{\frac{1}{3}}$

	Principal Method	**Exact keystroke sequence**	**Answer**
1	Power button: [x^y] Fraction button: [$a^b/_c$]	4913 [x^y] [(] 1 [$a^b/_c$] 3 [)] [=]	17
2	Power button: [x^y] Division	4913 [x^y] [(] 1 [÷] 3 [)] [=]	17
3	Root button: [$\sqrt[x]{\ }$]	3 [$\sqrt[x]{\ }$] 4913 [=]	17

Note that the brackets in both of the first two methods are essential. If you omit these brackets, you get a different answer (which is incorrect).

Example 4

Simplify $16^{\frac{3}{4}}$

	Principal Method	Exact keystroke sequence	Answer
1	Power button: $\boxed{x^y}$ Fraction button: $\boxed{a^b/_c}$	16 $\boxed{x^y}$ $\boxed{(}$ 3 $\boxed{a^b/_c}$ 4 $\boxed{)}$ $\boxed{=}$	8
2	Power button: $\boxed{x^y}$ Division	16 $\boxed{x^y}$ $\boxed{(}$ 3 $\boxed{\div}$ 4 $\boxed{)}$ $\boxed{=}$	8
3	Power button: $\boxed{x^y}$	16 $\boxed{x^y}$ 0.75 $\boxed{=}$	8

Again, note that the brackets in the first two methods are essential. The third method is not always available. For instance, if we needed to find $128^{\frac{3}{7}}$, then we cannot replace the fraction $\frac{3}{7}$ by a decimal. This is because the decimal version of $\frac{3}{7}$ recurs with a repeating block of six digits: 0.428571428571... If we try and use a decimal version of $\frac{3}{7}$, it will only be an approximation.

Fractional Powers of a Fraction

The above ideas can be extended to find a fractional power of a fraction. This is largely revision.

Example 5

Simplify:

(a) $\left(\frac{16}{81}\right)^{\frac{3}{4}}$ **(b)** $\left(\frac{16}{81}\right)^{-\frac{3}{4}}$

(a) As we did earlier, reverse the order of the two operations, i.e. do the powers first and then the division:

$$\left(\frac{16}{81}\right)^{\frac{3}{4}} = \frac{16^{\frac{3}{4}}}{81^{\frac{3}{4}}}$$

Recall that the four means a fourth root and the three means a cube.

$16^{\frac{3}{4}} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8$, and similarly $81^{\frac{3}{4}} = \left(\sqrt[4]{81}\right)^3 = 3^3 = 27$.

The final answer is therefore $\frac{8}{27}$.

(b) The only difference between (a) and (b) is the minus sign. The minus sign means that we do the reciprocal of the answer for (a). Recall that to do the reciprocal of a fraction, we swap the top and bottom. The answer to (b) is therefore $\frac{27}{8}$, which we could also write as the mixed number $3\frac{3}{8}$.

Finally, a reminder that it is often easier to work with improper fractions rather than mixed numbers.

Example 6

Simplify:

(a) $\left(2\frac{1}{4}\right)^{1\frac{1}{2}}$ **(b)** $\left(2\frac{1}{4}\right)^{-1\frac{1}{2}}$

(a) Change both the mixed numbers into improper fractions: $2\frac{1}{4} = \frac{9}{4}$ and $1\frac{1}{2} = \frac{3}{2}$.

So $\left(2\frac{1}{4}\right)^{1\frac{1}{2}} = \left(\frac{9}{4}\right)^{\frac{3}{2}}$.

Reverse the order of the two operations (power and division): $\frac{9^{\frac{3}{2}}}{4^{\frac{3}{2}}}$.

The two means square root, and the three means cube: $\frac{\left(\sqrt{9}\right)^3}{\left(\sqrt{4}\right)^3} = \frac{3^3}{2^3} = \frac{27}{8}$.

We could rewrite this final answer as $3\frac{3}{8}$ if required.

(b) The minus sign means do the reciprocal. It is best to work with the improper fraction $\frac{27}{8}$. The reciprocal of this is $\frac{8}{27}$.

Note that it is possible to double-check the answers to Example 6 using a calculator, if you are careful. However, your calculator may require some persuasion.

(2 [$a^{b}/_{c}$] 1 [$a^{b}/_{c}$] 4) x^y (1 [$a^{b}/_{c}$] 1 [$a^{b}/_{c}$] 2) =

The brackets are again essential. Also, the above sequence performed on a modern Casio generates a **decimal** answer. So you need to persuade the calculator to convert the decimal into a fraction by then pressing the fraction button [$a\,{}^{b}\!/_{c}$] again. This gives the answer in the form of the mixed number $3\frac{3}{8}$. If required, we could then ask the calculator to convert this to the improper fraction $\frac{27}{8}$ by pressing [SHIFT] followed by [$a\,{}^{b}\!/_{c}$].

Activity 5

1. Copy and complete the following table.

Fractional Index form	Root form	Simplest possible form
$16^{\frac{1}{2}}$	$\sqrt{16}$	4
$100^{\frac{1}{2}}$		
	$\sqrt[3]{27}$	
	$\sqrt[5]{?}$	2
$(?)^{\frac{1}{4}}$		3
$125^{?}$		5

2. Without using a calculator, simplify the following:

   (a) $16^{\frac{1}{2}}$ (b) $16^{\frac{3}{2}}$ (c) $32^{\frac{1}{5}}$ (d) $32^{\frac{4}{5}}$

   (e) $27^{\frac{1}{3}}$ (f) $27^{\frac{2}{3}}$

3. Without using a calculator, simplify the following:

   (a) $64^{\frac{1}{3}}$ (b) $64^{-\frac{1}{3}}$ (c) $32^{\frac{3}{5}}$ (d) $32^{-\frac{3}{5}}$

   (e) $10000^{\frac{3}{4}}$ (f) $10000^{-\frac{3}{4}}$

4. Use a calculator to simplify the following (write each answer as a fraction):

   (a) $300763^{\frac{1}{3}}$ (b) $7776^{\frac{2}{5}}$ (c) $4096^{\frac{11}{12}}$ (d) $49^{-\frac{1}{2}}$ (e) $2401^{-\frac{3}{4}}$

5. Without using a calculator, simplify the following (write each answer as a fraction):

   (a) $\left(\frac{8}{27}\right)^{\frac{1}{3}}$ (b) $\left(\frac{8}{27}\right)^{-\frac{1}{3}}$ (c) $\left(\frac{8}{27}\right)^{\frac{2}{3}}$ (d) $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$

6. Without using a calculator, simplify the following (write each answer as a fraction):

   (a) $\left(5\frac{1}{16}\right)^{1\frac{1}{4}}$ (b) $\left(5\frac{1}{16}\right)^{-1\frac{1}{4}}$

7. Use a calculator to simplify the following (write each answer as a fraction):

   (a) $\left(2\frac{1526}{3125}\right)^{-\frac{3}{5}}$ (b) $\left(2\frac{93}{125}\right)^{1\frac{1}{3}}$

Proportionality

The probability of lung cancer is proportional to how much you smoke.

The amount of radioactivity in a piece of material is inversely proportional to the time.

It is the job of mathematicians and statisticians to analyse and use statements such as these. Many of these problems are complex but we will only look at the simple basic techniques.

Direct Variation and Inverse Variation

We sometimes say that two quantities, such as x and y, are 'in direct proportion'. The mathematical meaning is the same as the everyday meaning. If one quantity doubles, for instance, then so does the other. If one quantity becomes halved, then so does the other. 'Inverse proportion' is a bit more technical. If x and y, are 'inversely proportional', then if one quantity doubles, then the other halves.

However, the important part of this topic is what you do, which is almost always as follows:

- Write an **equation** that describes the situation. This will involve a 'constant' *k*.
- Use the information provided to find the value of *k*.
- Now do whatever else the question requires.

Writing the Equation

We are often presented with a sentence involving the word 'proportional'. Replace the phrase 'proportional to' by an equals sign. Put the letter *k* immediately after the equals sign. This means that *k* **multiplies** the second quantity. Study the following table.

Sentence	**Equation**
y is proportional to *x*	$y = kx$
p is proportional to *t*	$p = kt$
y is proportional to the square of *t*	$y = kt^2$
V is proportional to the cube of *r*	$V = kr^3$
r is proportional to the square root of *A*	$r = k\sqrt{A}$

If the sentence includes the phrase 'inversely proportional', then the right-hand side of the equation becomes $\frac{k}{\text{second quantity}}$. Study the following table.

Sentence	**Equation**
y is inversely proportional to *x*	$y = \frac{k}{x}$
t is inversely proportional to *q*	$t = \frac{k}{q}$
y is inversely proportional to the square of *r*	$y = \frac{k}{r^2}$
P is inversely proportional to the cube root of *r*	$P = \frac{k}{\sqrt[3]{r}}$

Instead of a sentence, we are sometimes presented with an algebraic alternative involving the symbol '$\propto$'. This means 'proportional to'. We replace this symbol by an equals sign, and also insert the letter *k* as before. Study the following table.

Statement using '$\propto$'	Equation
$y \propto x$	$y = kx$
$x \propto t$	$x = kt$
$A \propto r^2$	$A = kr^2$
$f \propto \sqrt{T}$	$f = k\sqrt{T}$
$y \propto \frac{1}{x}$	$y = \frac{k}{x}$
$F \propto \frac{1}{r^2}$	$F = \frac{k}{r^2}$
$f \propto \frac{1}{\sqrt{m}}$	$f = \frac{k}{\sqrt{m}}$

Finding the value of *k*

We are usually given a pair of numbers which we substitute into the equation. With a bit of rearrangement, it is possible to find the value of *k*.

Example 1

Find the value of *k* in each situation below:

	Equation	Information provided
(a)	$y = kx$	*y* is 28 when *x* is 7.
(b)	$r = k\sqrt{A}$	*r* is 4 when *A* is 64.
(c)	$y = \frac{k}{x}$	*y* is 9 when *x* is 12.
(d)	$y = \frac{k}{r^2}$	*y* is 7 when *r* is 5.

(a) Substitute the given values in the equation: $28 = k \times 7$.
Rearrange: move the 7 to the left-hand side, changing the multiplication to division: $\frac{28}{7} = k$, so that *k* is 4.

(b) Substitute the given values in the equation: $4 = k \times \sqrt{64}$. But $\sqrt{64} = 8$, so we have $4 = k \times 8$. Move the 8 to the other side: $\frac{4}{8} = k$. We can now cancel to find the simplified value of $k = \frac{1}{2}$.

(c) Substitute the given values in the equation: $9 = \frac{k}{12}$. Now move the 12 to the left-hand side, changing the division into multiplication: $9 \times 12 = k$. So k is 108.

(d) Substitute the given values in the equation: $7 = \frac{k}{5^2}$. But $5^2 = 25$. Move this to the left-hand side: $7 \times 25 = k$. So k is 175.

The Full Routine

We are now ready for the full routine required by IGCSE questions on this topic. Note that the same three stages are always required, but that the question usually only mentions the third stage.

Example 2

(a) It is given that y is proportional to the square of x and that y is 144 when x is 6. Find the value of y when x is 7.

(b) It is given that R is inversely proportional to I, and that R is 14 when I is 18. Find the value of R when I is 42.

(c) It is given that P is inversely proportional to the square root of V, and that P is 63 when V is 25. Find the value of P when V is 100.

(a) Write the equation: $y = kx^2$. Substitute the known values: $144 = k \times 6^2$. Rearrange: $k = \frac{144}{6^2} = 4$. If we now replace k with 4, the equation becomes: $y = 4x^2$. We require the value of y when x is 7, so substitute $x = 7$ in the new version of the equation: $y = 4 \times 7^2 = 196$.

(b) Write the equation: $R = \frac{k}{I}$. Substitute the known values: $14 = \frac{k}{18}$. Rearrange to give $14 \times 18 = k$, so that we now know that k is 252. The equation is therefore $R = \frac{252}{I}$. When I is 42, R is given by $R = \frac{252}{42} = 6$.

(c) Write the equation: $P = \frac{k}{\sqrt{V}}$. Substitute the known values: $63 = \frac{k}{\sqrt{25}}$. Rearrange to give: $63 \times \sqrt{25} = k$, so that k must be $63 \times 5 = 315$. The equation is therefore $P = \frac{315}{\sqrt{V}}$. When V is 100, the corresponding value of P is given by $P = \frac{315}{\sqrt{100}} = \frac{315}{10} = 31.5$.

Example 3

It is given that $A \propto r^2$ and that A = 102.07 when r = 5.7. Use a calculator to find (correct to four significant figures):

(a) the value of A when r is 5.0
(b) the value of r when A = 100.

The equation is $A = kr^2$. Substitute the known values: $102.07 = k \times 5.7^2$. Rearrange to give $k = \frac{102.07}{5.7^2} = 3.14158...$

(a) Substituting r = 5.0, the required value of A is $3.14158 \times 5.0^2 = 78.54$ (correct to four significant figures).

(b) We need to rearrange the equation $A = kr^2$ to make r the subject. The new equation is $r = \sqrt{\frac{A}{k}}$. Substituting A = 100 and k = 3.14158, we obtain $r = \sqrt{\frac{100}{3.14158}} = 5.642$ (correct to four significant figures).

Activity 6

1. For each of the following statements, write down the corresponding equation:

(a) h is proportional to m
(b) E is proportional to the square of v
(c) P is proportional to the square root of V
(d) I is inversely proportional to W
(e) E is inversely proportional to the square of d
(f) $q \propto r$
(g) $w \propto s^3$
(h) $a \propto \frac{1}{\sqrt{b}}$

Questions 2 – 5: without calculator.

2 It is given that y is proportional to x and that y is 8 when x is 12. Find the value of y when x is 15.

3 It is given that y is inversely proportional to x and that y is 6 when x is 7. Find the value of y when x is 14.

4 It is given that f is proportional to the square root of T and that f is 220 when T is 400. Find the value of f when T is 625.

5 It is given that F is inversely proportional to the square of r and that F is 8 when r is 5. Find the value of F when r is 10.

Questions 6 – 7: with calculator

6 It is given that $V \propto r^3$ and that $V = 524$ when $r = 3.8$. Find the value of V, correct to three significant figures, when r is 4.1 .

7 It is given that $p \propto \frac{1}{\sqrt{t}}$, and that p is 17.2 when t is 8.5 .

Find the value of p when t is 10.0 (correct to three significant figures). Also find the value of t when p is 20.0 (correct to three significant figures).

Activity 7

The volume V of a cylinder is directly proportional to the square of the base radius r and the height h.

(a) Write down an expression connecting V with r and h.

(b) If h remains unchanged, but r becomes three times as long, how will V change?

(c) How will V change if both h and r are trebled?

Activity 8

Here is an example from Astronomy:

The time T for a planet to go round the Sun once is directly proportional to the cube of its distance (D) from the Sun.

For the Earth, $T = 1$ year and D is 1.5×10^8 km. Calculate how long it takes the planet Mars to orbit the Sun if its distance is 2.3×10^8 km. Give your answer correct to 3 significant figures.

Suggested Answers to Activities

Activity One

(a) $3\sqrt{2}$ (b) $2\sqrt{5}$ (c) $3\sqrt{3}$ (d) $4\sqrt{2}$

(e) $6\sqrt{2}$ (f) $4\sqrt{5}$ (g) $7\sqrt{2}$ (h) $6\sqrt{6}$

Activity Two

(a) $\frac{\sqrt{10}}{10}$

(b) $\frac{3\sqrt{2}}{2}$

(c) $\frac{\sqrt{15}}{10}$

Activity Three

1. **£900** = £1,317.69 ÷ $(1.1)^4$
2. **£2,500** = £3,345.56 ÷ $(1.06)^5$

Activity Four

1.

Original	%	Increase or	Multiplier	Final

	amount		decrease?		amount
a	£100	2.7	Increase	**1.027**	**£102.70**
b	£6480	3.82	Decrease	**0.9618**	**£6232.46**
c	£792	12.6	Increase	**1.126**	**£891.79**
d	£5000	8.47	Decrease	**0.9153**	**£4576.50**
e	**£623.39**	1.9	Increase	**1.019**	£635.23
f	**£7331.01**	4.25	Decrease	**0.9575**	£7019.44
g	**£70.68**	5.83	Increase	**1.0583**	£74.80
h	**£3855.66**	14.6	Decrease	**0.854**	£3292.73

2. (a) £3062.61 (b) £4081.49
3. £10174.50
4. £993.60
5. £4995

Activity Five

1

Fractional Index form	Root form	Simplest possible form
$16^{\frac{1}{2}}$	$\sqrt{16}$	4
$100^{\frac{1}{2}}$	$\sqrt{100}$	10
$27^{\frac{1}{3}}$	$\sqrt[3]{27}$	3
$32^{\frac{1}{5}}$	$\sqrt[5]{32}$	2
$81^{\frac{1}{4}}$	$\sqrt[4]{81}$	3
$125^{\frac{1}{3}}$	$\sqrt[3]{125}$	5

2 (a) 4
(b) 64
(c) 2
(d) 16
(e) 3
(f) 9

3 (a) 4
(b) $\frac{1}{4}$
(c) 8

(d) $\frac{1}{8}$
(e) 1000
(f) $\frac{1}{1000}$

4 (a) 67
(b) 36
(c) 2048
(d) $\frac{1}{7}$
(e) $\frac{1}{343}$

5 (a) $\frac{2}{3}$
(b) $\frac{3}{2} = 1\frac{1}{2}$
(c) $\frac{4}{9}$
(d) $\frac{9}{4} = 2\frac{1}{4}$

6 (a) $\frac{243}{32} = 7\frac{19}{32}$
(b) $\frac{32}{243}$

7 (a) $\frac{125}{216}$
(b) $\frac{2401}{625} = 3\frac{526}{625}$

Activity Six

1 k is a constant in each part (a) – (h).

(a) $h = km$ (b) $E = kv^2$ (c) $P = k\sqrt{V}$ (d) $I = \frac{k}{W}$

(e) $E = \frac{k}{d^2}$ (f) $q = kr$ (g) $w = ks^3$ (h) $a = \frac{k}{\sqrt{b}}$

2 $y = 10$
3 $y = 3$
4 $f = 275$
5 $f = 2$
6 $V = 658$
7 $p = 15.9$ $t = 6.29$ (3 s. f.)

Activity Seven

(a) $V \propto r^2h$

(b) Since h remains unchanged, V will be proportional to the square of r.
So if r is trebled then V will become 9 times as large.

(c) Since h is trebled, V will become three times bigger because of this.
It will become 9 times bigger because of the radius.
So the total change is $3 \times 9 = 27$

Activity Eight

$T \propto D^3$

So $T = kD^3$

For Earth: $1 = k \times (1.5 \times 10^8)^3$

So $k = \dfrac{1}{(1.5 \times 10^8)^3} = \dfrac{1}{(1.5)^3 \times 10^{24}}$

For Mars:

$$T = \frac{1}{(1.5)^3 \times 10^{24}} \times (2.3 \times 10^8)^3$$

$$= \frac{(2.3)^3}{(1.5)^3} = 3.61 \text{ years.}$$